Norm of bounded linear transformations.

Examples.

Example 1: Let N and N' be normed linear spaces and let T be a linear transformation of N into N'. Then T is continuous either at every point of N or at no point of N.

Solution: — Let x_1 and x_2 be any two points of N and suppose T is continuous at x_1. Then to each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x - x_1\| < \delta \Rightarrow \|T(x) - T(x_1)\| < \epsilon$$

Now $\|x - x_2\| < \delta \Rightarrow \|(x + x_1 - x_2) - x_1\| < \delta$

$$\Rightarrow \|T(x + x_1 - x_2) - T(x_1)\| < \epsilon$$

$$\Rightarrow \|T(x) + T(x_1) - T(x_2) - T(x_1)\| < \epsilon$$

(By linearity of T)

$$= \|T(x) - T(x_2)\| < \epsilon$$

This shows that T is continuous at x_2 as well. Since x_1, x_2 are arbitrary points, we have shown that if T is continuous at a particular point then it is continuous at all points.

Example-02: Let N and N' be normed linear spaces over the same scalar field and let T be a linear transformation of N into N'. Then T is bounded if and only if it is continuous.

Solution: — The 'only if' part — Let T be bounded so that there exists $M > 0$ such that

$$\|T(x)\| \le M\|x\|, \; x \in N \quad \text{——— (1)}$$

To show that T is continuous, let $x \in N$ be arbitrary. For any $\epsilon > 0$, we choose $\delta = \epsilon/M$. Then for all $y \in N$ such that $\|y - x\| < \delta$, we have

$$\|T(y) - T(x)\| = \|T(y - x)\|$$

$$\le M\|y - x\| \text{ by (1)} = M\frac{\epsilon}{M} = \epsilon$$

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Hence T is continuous at x. Since x is arbitrary, T is a continuous mapping.

If Part: Let T be continuous and suppose, if possible, T is not bounded i.e. there exists no $K>0$ such that

$$\|N(x)\| \le K\|x\| \text{ for all } x \in N.$$

Then for each positive integer n, there exists a point $x_n \in N$ such that $\|T(x_n)\| > n\|x_n\|$.

For each n, we let $y_n = \frac{x_n}{n\|x_n\|}$

Then $\|y_n\| = \frac{1}{n} \to 0$ as $n \to \infty$ which implies that $y_n \to 0$ as $n \to \infty$. but for every n,

$$\|T(y_n)\| = \left\|\left(\frac{x_n}{n\|x_n\|}\right)\right\| = \left\|\frac{1}{n\|x_n\|} T(x_n)\right\|$$

$$= \frac{1}{n\|x_n\|} \|T(x_n)\|$$

and since $\|T(x_n)\| > n\|x_n\|$, we conclude that

$$\|T(y_n)\| > 1.$$

This implies that $T(y_n)$ does not tend to $0 \;(= T(0))$ as $n \to \infty$. Thus the sequence $\langle y_n \rangle$ converges to 0 but the sequence $\langle T(y_n) \rangle$ does not converge to $T(0)$. Hence T is not continuous at 0 which is a contradiction. Hence T must be bounded.

Example (3):- Let M be a closed linear sub-space of a normed linear space N and let ϕ be the natural mapping (homomorphism of N onto N/M defined by $\phi(x) = x+M$ show that ϕ is a continuous (or bounded) linear transformation for which $\|\phi\| \leq 1$.

Solution:— Since M is closed, N/M is a normed linear space with the norm of a coset $x+M$ in N/M defined by

$$\|x+M\| = \inf\{\|x+m\| : m \in M\}.$$

ϕ is linear.

Let x, y be any two elements of N and α, β any scalars. Then

$$\phi(\alpha x+\beta y) = (\alpha x+\beta y)+M = (\alpha x+M)+(\beta y+M)$$
$$= \alpha(x+M) + \beta(y+M) = \alpha\phi(x)+\beta\phi(y)$$

is continuous

$$\|\phi(x)\| = \|x+M\| = \inf\{\|x+M\| : m \in M\}$$
$$\leq \|x+m\| \ \forall\ m \in M.$$

In particular for $m=0$, we have

$$\|\phi(x)\| \leq \|x\| = 1 \cdot \|x\| \ \forall\ x \in N \quad \text{—— (1)}$$

It follows that ϕ is bounded by the bound 1 and consequently ϕ is continuous. Further

$$\|\phi\| = \sup\{\|\phi(x)\| : x \in N, \|x\| \leq 1\}.$$

$$\leq \sup\{\|x\| : x \in N \ \|x\| \leq 1\} \text{ by (1)} = 1.$$

Thus $\|\phi\| \leq 1$.

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